

Hamilton Cycles in Double Generalized Petersen Graphs

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Abstract

Watkins (1969) first introduced the generalized Petersen graphs (GPGs) by modifying Petersen graph. Zhou and Feng (2012) modified GPGs and introduced the double generalized Petersen graphs (DGPGs). Kutnar and Petecki (2016) proved that DGPGs are Hamiltonian in special cases and conjectured that all DGPGs are Hamiltonian. In this paper, we construct Hamilton cycles in all DGPGs.

Keywords: Hamilton cycle, Double generalized Petersen graph

1. Introduction

In [1] Watkins (1969) first introduced the GPGs to discover Tait coloring of the graphs and Castagna and Prins (1972) proved Watkins' conjecture about GPGs in [2]. After they introduced GPGs, some properties of GPGs have been studied. For instance, Alspach (1983) determined which GPGs have a Hamilton cycle in [3]. Fu, Yang and Jiang (2009) studied the domination number of GPGs in [4].

Now we define double generalized Petersen graphs $DP(n, t)$ (DGPG for short) as follows.

Definition 1. Let n and t be integers that satisfy $n \geq 3$ and $2 \leq 2t < n$. The double generalized Petersen graph $DP(n, t)$ is an undirected simple graph with vertex set V and edge set E , where

$$V = \{x_i, u_i, v_i, y_i \mid i \in \mathbb{Z}_n\},$$
$$E = \{x_i x_{i+1}, y_i y_{i+1}, x_i u_i, y_i v_i, u_i v_{i+t}, v_i u_{i+t} \mid i \in \mathbb{Z}_n\}$$

Note that \mathbb{Z}_n denotes a set of integers $\mathbb{Z}/n\mathbb{Z}$ throughout this paper.

Zhou and Feng (2012) first introduced the double generalized Petersen graphs by modifying the generalized Petersen graphs in [5]. In [6], Zhou and Feng (2014) determined all non-Cayley vertex-transitive graphs and all vertex-transitive graphs among DGPGs. From their result, Kutnar and Petecki (2016) gave the complete classification of automorphism groups of DGPGs in [7]. They also proved that $DP(n, t)$ is Hamiltonian if n is even or n is odd and the greatest common divisor of n and t equals to 1 in [7].

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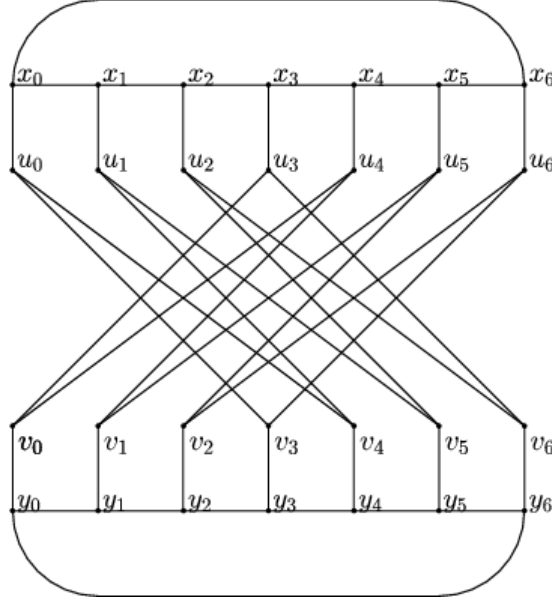


Figure 1: DP(7, 3)

In addition, a computer-assisted search verified that $DP(n, t)$ have Hamilton cycles for all $n \leq 31$ and they conjectured that all $DP(n, t)$ are Hamiltonian. This paper gives the following theorem.

Theorem 2. *All $DP(n, t)$ are Hamiltonian.*

2. Preliminaries

As mentioned in the previous section, \mathbb{Z}_n denotes a set of integers $\mathbb{Z}/n\mathbb{Z}$. A sequence of vertices $w_0 w_1 w_2 \dots w_n$ denotes a path in a graph. A path whose end points are the same vertex is called a cycle. $V(G)$ denotes the vertex set of a graph G . Let G be an arbitrary subgraph of $DP(n, t)$. We define functions V_x, V_y, V_u, V_v as follows.

$$V_x(G) = V(G) \cap \{x_i \mid i \in \mathbb{Z}_n\}$$

$$V_y(G) = V(G) \cap \{y_i \mid i \in \mathbb{Z}_n\}$$

$$V_u(G) = V(G) \cap \{u_i \mid i \in \mathbb{Z}_n\}$$

$$V_v(G) = V(G) \cap \{v_i \mid i \in \mathbb{Z}_n\}$$

3. The construction of Hamilton cycles in $\text{DP}(n, t)$

We assume that n is even. In this case, Kutnar and Petecki showed that all $\text{DP}(n, t)$ are Hamiltonian in [7]. Observe that there exist paths X_i for all $i \in \mathbb{Z}_{n/2}$.

$$X_i : u_{2i}x_{2i}x_{2i+1}u_{2i+1}v_{2i+1-t}y_{2i+1-t}y_{2i+2-t}v_{2i+2-t}u_{2(i+1)}$$

Joining all of the paths gives a Hamilton cycle in $\text{DP}(n, t)$.

We assume that n is odd. Let $2k + 1$ be the greatest common divisor of n and t . In order to construct a Hamilton cycle in $\text{DP}(n, t)$, we define paths P_i, Q_i, R_i, S_i for all $i \in \mathbb{Z}_{2k+1}$.

$$\begin{aligned} P_i &: u_{a_i+t}x_{a_i+t}x_{a_i+t+1}x_{a_i+t+2} \cdots x_{a_i+2+t-1}u_{a_i+2+t-1} \\ Q_i &: v_{a_i}y_{a_i}y_{a_i+1}y_{a_i+2} \cdots y_{a_i+2-1}v_{a_i+2-1} \\ R_i &: u_{a_{i+1}+t-1}v_{a_{i+1}+2t-1}u_{a_{i+1}+3t-1} \cdots v_{a_i} \\ S_i &: v_{a_{i+1}-1}u_{a_{i+1}-t-1}v_{a_{i+1}-2t-1} \cdots u_{a_i+t} \end{aligned}$$

where $a_0, a_1, a_2, \dots, a_{2k} \in \mathbb{Z}_{2k+1}$ satisfy the following conditions

$$\begin{aligned} \forall i \in \mathbb{Z}_{2k+1}, a_i &\equiv i \pmod{2k+1} \\ 0 \leq a_0 < a_2 < a_4 < \cdots < a_{2k} < a_1 < a_3 < a_5 < \cdots < a_{2k-1} < n \end{aligned}$$

For instance, if $a_0 = 0, a_2 = 2, a_4 = 4, \dots, a_{2k} = 2k, a_1 = 2k + 2, a_3 = 2k + 4, a_5 = 2k + 6, \dots, a_{2k-1} = 4k$, the above conditions are met. Joining the paths in the following way gives a Hamilton cycle in $\text{DP}(n, t)$.

$$\begin{aligned} &((S_0 - P_0) - (R_1 - Q_1) - (S_2 - P_2) - (R_3 - Q_3) - \cdots \\ &\quad \cdots - (R_{2k-1} - Q_{2k-1}) - (S_{2k} - P_{2k})) - \\ &-((R_0 - Q_0) - (S_1 - P_1) - (R_2 - Q_2) - (S_3 - P_3) - \cdots \\ &\quad \cdots - (S_{2k-1} - P_{2k-1}) - (R_{2k} - Q_{2k})) \end{aligned}$$

An example of a Hamilton cycle in $\text{DP}(n, t)$ is shown in Figure 2.

4. Proof of Theorem 2

25 In this section, we prove that the cycle described in the previous section contains all vertices of $\text{DP}(n, t)$ exactly once for all odd integers $n \geq 3$. Let G be $\text{DP}(n, t)$ and $2k + 1$ be the greatest common divisor of n and t .

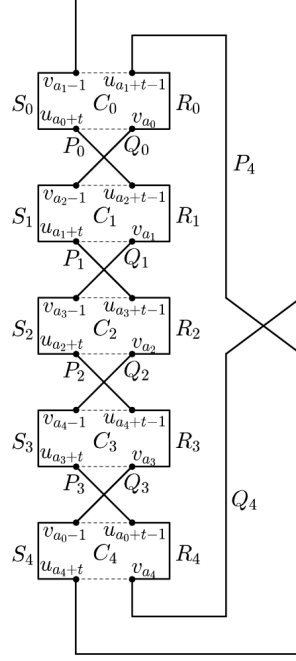


Figure 2: A Hamiltonian cycle in $DP(n, t)$ ($2k + 1 = 5$)

Firstly, we prove that paths Q_0, Q_1, \dots, Q_{2k} contain all of $V_y(G)$.

$$\begin{aligned}
 \bigcup_{i=0}^{2k} V_y(Q_i) &= \left(\bigcup_{i=0}^k V_y(Q_{2i}) \right) \cup \left(\bigcup_{i=0}^{k-1} V_y(Q_{2i+1}) \right) \\
 &= (\{y_{a_0}, y_{a_0+1}, \dots, y_{a_2-1}\} \cup \\
 &\quad \{y_{a_2}, y_{a_2+1}, \dots, y_{a_4-1}\} \cup \\
 &\quad \{y_{a_4}, y_{a_4+1}, \dots, y_{a_6-1}\} \cup \\
 &\quad \vdots \\
 &\quad \{y_{a_{2k}}, y_{a_{2k}+1}, \dots, y_{a_{1}-1}\}) \cup \\
 &\quad (\{y_{a_1}, y_{a_1+1}, \dots, y_{a_3-1}\} \cup \\
 &\quad \{y_{a_3}, y_{a_3+1}, \dots, y_{a_5-1}\} \cup \\
 &\quad \{y_{a_5}, y_{a_5+1}, \dots, y_{a_7-1}\} \cup \\
 &\quad \vdots \\
 &\quad \{y_{a_{2k-1}}, y_{a_{2k-1}+1}, \dots, y_{a_0-1}\}) \\
 &= \{y_m \mid m \in \mathbb{Z}_n\} \\
 &= V_y(G)
 \end{aligned}$$

According to the above equation and the definitions of P_i and Q_i , we can prove that

paths P_0, P_1, \dots, P_{2k} contain all of $V_x(G)$.

Secondly, we will prove that paths $R_0, R_1, \dots, R_{2k}, S_0, S_1, \dots, S_{2k}$ contain all of $V_u(G) \cup V_v(G)$. We define cycles C_i in $\text{DP}(n, t)$ for all $i \in \mathbb{Z}_{2k+1}$.

$$C_i: u_i v_{i+t} u_{i+2t} v_{i+3t} \cdots u_{i+(p-1)t} v_i u_{i+t} v_{i+2t} u_{i+3t} \cdots v_{i+(p-1)t} u_i$$

- 30 Note that odd integers p and q satisfy $n = p(2k+1)$ and $t = q(2k+1)$. For all $i \in \mathbb{Z}_{2k+1}$, C_i consists of paths $D_i: u_i v_{i+t} u_{i+2t} v_{i+3t} \cdots u_{i+(p-1)t}$ and $E_i: v_i u_{i+t} v_{i+2t} u_{i+3t} \cdots v_{i+(p-1)t}$. Since p is odd, the last vertex of D_i is not $v_{i+(p-1)t}$ but $u_{i+(p-1)t}$. By symmetry, the last vertex of E_i is $v_{i+(p-1)t}$. In addition, $u_{i+(p-1)t}$ and $v_{i+(p-1)t}$ are respectively adjacent to v_i and u_i since $pt = pq(2k+1)$ is a multiple of n . Observe that $u_i, u_{i+t}, u_{i+2t}, u_{i+3t}, \dots, u_{i+(p-1)t}$ contain no two same vertices since pt is the least common multiple of n and t . Hence
- 35 $v_i, v_{i+t}, v_{i+2t}, v_{i+3t}, \dots, v_{i+(p-1)t}$ also contain no two same vertices.

We show that cycles C_0, C_1, \dots, C_{2k} contain all of $V_u(G) \cup V_v(G)$.

$$\begin{aligned} \bigcup_{i=0}^{2k} V_u(C_i) &= \bigcup_{i=0}^{2k} \{u_{i+jt} \mid 0 \leq j < p\} \\ &= \bigcup_{i=0}^{2k} \{u_{i+jq(2k+1)} \mid 0 \leq j < p\} \\ &= \bigcup_{i=0}^{2k} \{u_{i+j(2k+1)} \mid 0 \leq j < p\} \\ &= \bigcup_{i=0}^{2k} \{u_m \mid m \in \mathbb{Z}_n, m \equiv i \pmod{2k+1}\} \\ &= \{u_m \mid m \in \mathbb{Z}_n\} \end{aligned}$$

By symmetry, we have

$$\begin{aligned} \left(\bigcup_{i=0}^{2k} V_u(C_i) \right) \cup \left(\bigcup_{i=0}^{2k} V_v(C_i) \right) &= \{u_m \mid m \in \mathbb{Z}_n\} \cup \{v_m \mid m \in \mathbb{Z}_n\} \\ &= V_u(G) \cup V_v(G) \end{aligned}$$

- Observe that both R_i and S_i are subgraphs of C_i . For all $i \in \mathbb{Z}_{2k+1}$, R_i and Q_i share no vertex and contain all vertices in C_i since the first vertex of R_i and the last vertex of R_i are respectively adjacent to the first vertex of S_i and the last vertex of S_i . Therefore,
- 40 paths $R_0, R_1, \dots, R_{2k}, S_0, S_1, \dots, S_{2k}$ contain all of $V_u(G) \cup V_v(G)$. This completes the proof of Theorem 2. ■

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